Quillen's Higher algebraic K-theory

Ref: Quillen : Higher algebraic K-theory I.
Homotopical algebra Sec 2.1 - 2.5.

Higher algebraic K-theory assigns any ring A a sequence of abelian groups $K_i(A)$ in a functorial way. We define
$$K_i(A) = \pi_i(K(A)) \text{ for certain topological space}$$
$K(A)$ constructed from A.

Step 1. : Construct a small category $\underline{K}(A)$

Step 2: Take the geometric realization $B\underline{K}(A) = K(A)$.

Recall K_2.

Steinberg group $St_n(A)$ as a group generated by symbols $x_{ij}(a)$ for $1 \le i \ne j \le n$, $a \in A$ modulo equivalence relations.

1) $x_{ij}(a) \cdot x_{ij}(b) = x_{ij}(a+b)$

2) $[x_{ij}(a), x_{kl}(b)] = \begin{cases} 1 & j \ne k, i \ne l \\ x_{il}(ab) & j = k, i \ne l \\ x_{kj}(-ba) & j \ne k, i = l \end{cases}$

Recall elementary matrices $E_n = \langle e_{ij}(a) \rangle$ also satisfies above condition. Hence $\exists$ natural map $\varphi_n : St_n(A) \to GL_n(A)$.
which factored as $St_n(A) \to E_n(A) \hookrightarrow GL_n(A)$.

Let $St(A) = \operatorname{colim} St_n(A)$.

Recall $K_1(A) = \operatorname{coker}(\varphi)$ (since $E(A) = [E(A), E(A)]$
by Whitehead. $= [GL(A), GL(A)]$

Def: $K_2 = \ker \varphi$

So $\exists$ exact sequence of groups.

$$1 \to K_2(A) \to St(A) \xrightarrow{\varphi} GL(A) \longrightarrow K_1(A) \to 1.$$

THM (Steinberg) K_2 is abelian. In fact $K_2 = Z(St(A))$

Universal central extension.

Recall, a central extension of G by A is a s.e.s.

$$0 \to A \xrightarrow{i} \tilde{G} \xrightarrow{p} G \to 1$$

s.t. $i(A) \subseteq Z(\tilde{G})$

Lemma: $\exists$ a bijection $\left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{central ext. of } G \text{ by } A \end{array} \right\} \longleftrightarrow H^2(G, A)$.

Note: two central extension are equivalent if $\exists f: \tilde{G}_1 \to \tilde{G}_2$

s.t.

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, equal] & \tilde{G}_1 \arrow[r, "p_1"] \arrow[d, "f"] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A \arrow[r, "i_2"] & \tilde{G}_2 \arrow[r, "p_2"] & G \arrow[r] & 1 \end{tikzcd}$$

commutes.

For a group G, define a category $\underline{Ext}_C(G)$ with

$$ob(\underline{Ext}_C(G)) = \{ p: \tilde{G} \overset{p}{\twoheadrightarrow} G : \operatorname{Ker}(p) \subseteq Z(\tilde{G}) \}$$

$$\operatorname{Hom}_{\underline{Ext}_C(G)}(\tilde{G}_1, \tilde{G}_2) = \{ f: \tilde{G}_1 \to \tilde{G}_2 \mid p_2 \circ f = p_1 \}.$$

Def: The universal central extension is ~~an~~ the initial object in ~~the~~ $Ext_C(G)$.

THM. The universal central extension exists iff G is perfect.

THM (Recognition criterion) $\tilde{G} \to G$ a central extension, TFAE: 1) $\tilde{G}$ is universal

2) $\tilde{G}$ is perfect and every central extension of $\tilde{G}$ splits

3) $H_1(\tilde{G}, \mathbb{Z}) = H_2(\tilde{G}, \mathbb{Z}) = 0$.

THM (Kervaire, Steinberg) The natural exact sequence $1 \to K_2(A) \to St(A) \to E(A) \to 1$ is the universal central extension of $E(A)$. $\therefore$ $K_2(A) \cong H_2(E(A), \mathbb{Z})$

Def: $K_3(A) = H_3(St(A, \mathbb{Z})$.

Plus - construction.

Recall : A topological space X is acyclic if $\tilde{H}_i(X,\mathbb{Z})=0$ $\forall i\in\mathbb{Z}$.

If X is acyclic, we have : 1) X is connected

2) $\pi_1(X)$ is perfect, i.e. $\pi_1(X)=[\pi_1(X),\pi_1(X)]$

3) $H_2(\pi_1(X),\mathbb{Z})=0$

Recall: $F_f=\{(e,\gamma)\in(X,P)$: $f(e)=\gamma(1)$, $\gamma(0)=*\}$.

Def: Let X,Y be two based connected CW complex. A map $f:X\to Y$ is acyclic if its homotopy fiber F_f is acyclic.

Rmk: X is ~~acyclic~~ acyclic $\Leftrightarrow$ $X\to *$ is acyclic.

Prop: Sps f is acyclic, then 1) $f_*:\pi_1(X)\to\pi_1(Y)$ is surjective

2) $\operatorname{Ker}(f_*)$ is a perfect normal subgroup of $\pi_1(X)$.

DEF Let X be a based CW-cpx. Let $N\trianglelefteq\pi_1(X)$ be a perfect normal subgroup of $\pi_1(X)$. A +-construction on X relative to N is an acyclic map $f:X\to X_N^+$ s.t. $\operatorname{Ker}[f_*:\pi_1(X)\to\pi_1(X_N^+)]=N$.

Quillen showed that +-construction exists and is universal among all maps $g:X\to Y$ with $g_*(N)=\{1\}\subset\pi_1(Y)$.

$\Rightarrow$ X_N^+ is unique (up to homotopy).

Every group G has a unique largest perfect group $\Rightarrow$ denote this group $P(G)$, the perfect radical. Note, $P(G)\subset G$ is normal.

Def: We call +-construction ~~of~~ on X w.r.t. $P(\pi_1(X))$ the plus construction : $f: X \to X^+$.

Higher K-groups.

Let A be ~~the~~ a unital associative ring. define $K_i(A) = \pi_i(BGL(A)^+)$.

For $i=1$, $K_1(A) = \pi_1(BGL(A)^+) = \pi_1(BGL(A)) / P(\pi_1(BGL(A))$

(Whitehead lem) $= GL(A) / [GL(A), GL(A)]$

Hence K_i: Rings $\to$ Ab

Def: The K-theory of A is the space.

$$K(A) = K_0 \times BGL(A)^+ = \coprod_{K_0} BGL(A)^+$$

Hence $\pi_0(K(A)) = K_0(A)$

$\pi_i(K(A)) = K_i(A) \quad i \geq 1$

Prop. Quillen's def. of $K_2(A)$ agrees with Milnor-Steinberg.

Pf: WTS $\pi_2(BGL(A)^+) \simeq H_2(E(A), \mathbb{Z})$.

Lemma: Let $N \trianglelefteq G$ perfect normal. Consider $f: BG \to BG_N^+$, let F_f be the homotopy fibre. Then

1) $\pi_1(F_f)$ is isomorphic to the universal central extension of N

2) $\pi_2(BG_N^+) \simeq H_2(N, \mathbb{Z})$

Pf: We have homotopy fibration $F_f \to BG \to BG_N^+$.

Hence we have.

$$\cdots\ \pi_2(BG) \to \pi_2(BG_N^+) \to \pi_1(Ff) \to \pi_1(BG) \to \pi_1(BG_N^+) \to \pi_0(Ff)$$
$$\pi_2(BG) = \{1\}, \quad \pi_1(BG) = G, \quad \pi_1(BG_N^+) = G/N, \quad \pi_0(Ff) = \{1\}.$$

Therefore, we get a s.e.s.

$$1 \to \pi_2(BG_N^+) \to \pi_1(Ff) \to N \to 1$$

Ref Whitehead ← Element of homotopy theory IV, 3.5

By a property of homotopy fibration, $\underset{\text{normal}}{\pi_2(BG_N^+)} \subset Z(\pi_1(Ff))$.
By previous observation, $\pi_1(Ff)$ is a perfect subgroup.
So $H_1(\pi_1(Ff), \mathbb{Z}) = 0 = H_2(\pi_1(Ff), \mathbb{Z})) = 0$.
$\Rightarrow \pi_1(Ff) \simeq$ universal central extension of N.

THM (Hopf). Consider a presentation. $G \simeq F/R$, $\exists$ 2 natural central extensions.

1. $1 \to \frac{R}{[R,F]} \to \frac{F}{[R,F]} \to G \to 1$
2. $1 \to \frac{R \cap [F,F]}{[R,F]} \to \frac{[F,F]}{[R,F]} \to [G,G] \to 1$

If G perfect, the second one is the universal central extension and we have $H_2(G,\mathbb{Z}) \simeq \frac{R \cap [F,F]}{[R,F]}$.

Hence, we have ~~$\pi_2(BG)$~~ $\pi_2(BG_N^+) \simeq H_2(N, \mathbb{Z})$.
Now let $N = E(A) \subseteq GL(A) = G$, we get
$$K_2(A) = \pi_2(BGL(A)^+) = H_2(E(A, \mathbb{Z})). \quad \square$$

Lemma. Let $N \trianglelefteq G$ be a perfect normal subgroup,
$f: BG \to BG_N^+$ the corresponding + construction,
The we have the factorization.

$$\begin{tikzcd} BN \arrow[r] \arrow[dr] & BG \arrow[r] & BG_N^+ \\ & BN^+ \arrow[ur, dashed] & \end{tikzcd}$$

$\therefore$ BN^+ is ~~homo~~ homotopy equiv. to the universal covering of BG_N^+. Therefore, $\pi_i(BN^+) \cong \pi_i(BG_N^+)$ $i \geq 2$.

<u>Cor</u> $\forall$ ring A, $K_i(A) \cong \pi_i(BE(A)^+)$ $\forall i \geq 2$

Let $1 \to A \to \tilde{G} \to G \to 1$ be the universal central extension of G. So $G, \tilde{G}$ are perfect groups. Consider

$$\begin{tikzcd} BA \arrow[r] & B\tilde{G} \arrow[r] \arrow[dr] & BG \arrow[r] & BG^+ \\ & & B\tilde{G}^+ \arrow[ur, dashed] & \end{tikzcd}$$

We have a sequence $BA \to B\tilde{G}^+ \to BG^+$

This is in fact a homotopy fibration sequence.

$\pi_i(B\tilde{G}^+) = 0$ $i = 0, 1, 2$.

$\pi_i(B\tilde{G}^+) = \pi_i(BG^+)$ $i \geq 3$.

<u>Cor.</u> $\forall i \geq 3$, $\exists$ isomorphism $K_i(A) \simeq \pi_i(BSt(A)^+)$.

Hence $K_3(A) = H_3(St(A), \mathbb{Z})$.

Q - construction

1. Exact categories.

Def: An exact category is an additive categories $\mathcal{C}$ equipped w/ a class $\mathcal{E}$ of diagram of the form $(i,p) := [c' \overset{i}{\hookrightarrow} c \overset{p}{\twoheadrightarrow} c'']$. We call i admissible monic, call p admissible epic. $\mathcal{E}$ satisfies

E1) closed under isomorphisms.

E2) the set of admissible monics is close under composition and pushouts along arbitrary morphisms. i.e. $i: c \to c'$ adimissible monic, $f: c \to d$ arbitrary, then

$$\begin{tikzcd} c \arrow[r, "i"] \arrow[d, "f"'] & c' \arrow[d, dashed] \\ d \arrow[r, dashed, "\tilde{i}"] & d \times_c c' \end{tikzcd}$$

$\Rightarrow \tilde{i}$ also admissible monic.

E3) the set of admissible epic is closed under compositions and pull backs. i.e.

$$\begin{tikzcd} d \times^{c'}_{c} \arrow[r, dashed, "\tilde{p}"] \arrow[d, dashed] & d \arrow[d, "f"] \\ c \arrow[r, "p"] & c' \end{tikzcd}$$

$\Rightarrow \tilde{p}$ admissible epic.

E4) diagram of the form $[c \hookrightarrow c \oplus d \twoheadrightarrow d] \in \mathcal{E}$

E5) if $(i,p) \in \mathcal{E}$, $i = \mathrm{Ker}(p)$ $p = \mathrm{Coker}(i)$

Def: A functor $F: (\mathcal{C}, \mathcal{E}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{E}_{\mathcal{D}})$ between two exact categories are exact if $F(\mathcal{E}_C) = \mathcal{E}_D$.

2. K-group of an exact category.

Defn $K_0(\mathcal{C}, \mathcal{E}) := \mathbb{Z}[Iso(\mathcal{C})] \,/\, ([C] = [C'] + [C''] \text{ for } [C' \to C \to C''] \text{ in } \mathcal{E})$

Ex. Let A be an abelian categories, then A is exact with $\mathcal{E}$ being all short exact sequences. $\Rightarrow$ canonical exact structure on A.

Note, there is another exact structure on A. Consider $\mathcal{E} = \{$ all split exact sequence, i.e. $0 \to c \to c \oplus d \to d \to 0 \}$.

In general, $K_0(A, \mathcal{E}) \not\cong K_0(A, \mathcal{E}^{split})$.

Ex. Let A be a unital ring. Consider $\mathcal{C} = \mathbb{P}(A)$, the category of f.g. proj. (right) A-modules. Take $\mathcal{E}_{\mathbb{P}(A)}$ be all s.e.s. $\mathcal{C}$ is not abelian, but exact.

Then $K_0(A) := K_0(\mathbb{P}(A), \mathcal{E}_{\mathbb{P}})$.

Ex. X : quasi-projective scheme, let $\mathcal{C} = \underline{Vect}(X)$ category of vector bundles on X. $K^0(X) = K_0(\underline{Vect}(X))$

Note $K^0(X)$ is the Grothendieck ring.

$K_0(X)$ is the Grothendieck group defined by using coherent sheaves. In nice cases, $K_0(X) \cong K^0(X)$.

Ex. Let $\mathcal{C}$ be a skeletally small symmetric monoidal category. We can define the ~~relate~~ groupoid $\underline{Iso}(\mathcal{C})$ associated to $\mathcal{C}$.

Let $Iso(\mathcal{C}) = \pi_0(\underline{Iso}(\mathcal{C}))$, this is an abelian monoid. Define a group $K_0^{\otimes}(\mathcal{C})$ be the group completion of the monoid $S = Iso(\mathcal{C})$. i.e. $K_0^{\otimes}(\mathcal{C}) = S^{-1}S$.

Let $\mathcal{C}$ be an additive category, then it is naturally symmetric monoidal, with monoidal structure given by direct sum $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Consider K-group $K_0^{\oplus}(\mathcal{C})$. If $\mathcal{C}$ is also exact, we have $K_0(\mathcal{C}, \mathcal{E})$. In general they are not isomorphic. In fact, $K_0(\mathcal{C}, \mathcal{E})$ is naturally a quotient of $K_0^{\oplus}(\mathcal{C})$.

<u>Lem</u>. Let $\mathcal{B}$ be an ~~exact~~ exact subcategory of $\mathcal{C}$, s.t. $\mathcal{B}$ is closed under extension and cofinal. Then $K_0(\mathcal{B})$ is naturally a subgroup of $K_0(\mathcal{C})$

<u>Cor</u> $K_0(\mathcal{C}) / K_0(\mathcal{B}) \simeq K_0^{\oplus}(\mathcal{C}) / K_0^{\oplus}(\mathcal{B})$.

3. <u>Q - construction.</u>

Let $(\mathcal{C}, \mathcal{E})$ be skeletally small. Define a category $Q\mathcal{C}$. with.

1) $Ob(Q\mathcal{C}) = Ob(\mathcal{C})$.

2) $Hom_{Q\mathcal{C}}(C, C') = \left\{ C \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} C' \right\} / \sim$ called 'roof'

where p: admissible epic

i: adimissible monic

where $\sim$ is: two roofs $[C \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} C']$

$[C \overset{p'}{\twoheadleftarrow} d' \overset{i'}{\hookrightarrow} C']$

are equivalent if

$\exists$ an isomorphism $\varphi : d \xrightarrow{\sim} d'$ s.t.

$$\begin{tikzcd} c \arrow[d, equal] & d \arrow[l, two heads, "p"'] \arrow[r, hook, "i"] \arrow[d, "\varphi", "\sim"'] & c' \arrow[d, equal] \\ c & d' \arrow[l, two heads, "p'"'] \arrow[r, hook, "i'"] & c \end{tikzcd}$$

The composition of two roofs $[c \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} c']$ and $[c' \overset{p'}{\twoheadleftarrow} d' \overset{i'}{\hookrightarrow} c'']$ is the equivalence class of

$$\begin{tikzcd} & & d \times_{c'} d' \arrow[dl, "\tilde{p}'"'] \arrow[dr, "\tilde{i}"] & & \\ & d \arrow[dl, "p"'] \arrow[dr, hook, "i"] & & d' \arrow[dl, "p'"'] \arrow[dr, hook, "i'"] & \\ c & & c' & & c'' \end{tikzcd}$$

where $d \times_{c'} d'$ is pull-back of d' along i.

Rmk. There two distinguished type of roof.

$$\begin{tikzcd} & d \arrow[dl, equal] \arrow[dr, hook, "i"] & \\ d & & c' \end{tikzcd} \qquad \begin{tikzcd} & d \arrow[dl] \arrow[dr, equal] & \\ c' & & d \end{tikzcd}$$

which are exactly admissible monic/epic in $\mathcal{C}$.
We denote them by $[d \overset{i}{\hookrightarrow} c]$ and $[c \overset{p}{\twoheadleftarrow} d]$

Rmk. $[c \overset{p}{\twoheadleftarrow} d \xrightarrow{i} c'] = [c \twoheadleftarrow d] \circ [d \overset{i}{\hookrightarrow} c']$

Def: Let $c \in Ob(\mathcal{C})$. A ~~subobject~~ subobject of c is an equivalence class of monics $d \hookrightarrow c$ s.t. two monics are equivalent if they factored through each other. A subobject is admissible if it can be represented by admissible monic

Every morphism $[c \twoheadleftarrow d \hookrightarrow c']$ defines a unique admissible monic $d \hookrightarrow c'$. Fix representatives for each subobject. Then a morphism $c \to c'$ in $Q\mathcal{C}$ is given by an admissible subobject $d \hookrightarrow c'$ in c' and an admissible epic $d \twoheadrightarrow c$. Hence $\mathrm{Hom}_{Q\mathcal{C}}(0, c') = \{$admissible sub obj. of c' in $\mathcal{C}\}$.

Also isomorphisms in $Q\mathcal{C}$ corresponds to isomorphism in $\mathcal{C}$.

The K_0 group via Q-construction.

Let $\mathcal{C}$ be skeletally small exact

Note. 1) $Q\mathcal{C}$ is then skeletally small, so $BQ\mathcal{C}$ is well defined.

2) $BQ\mathcal{C}$ has an canonical base point given by $0 \in Ob(\mathcal{C})$.

3) $BQ\mathcal{C}$ is connected

THM $K_0(\mathcal{C}) \simeq \pi_1(BQ\mathcal{C})$, when $[C] \in K_0(\mathcal{C})$ corresponds to $[0 \hookrightarrow c \twoheadrightarrow 0] = [0 \hookrightarrow c] \circ [c \twoheadrightarrow 0]$.

Pf: Let T be the class of all morphisms $[0 \hookrightarrow a] \in Mor(Q\mathcal{C})$. $\Rightarrow$ T is a maximal tree. Then we have an presentation of $\pi_1(BQ\mathcal{C})$ (since $Q\mathcal{C}$ is small and connected):

$$\pi_1(BQ\mathcal{C}) = \langle [[Mor\, Q\mathcal{C}]] \rangle / S$$

where the normal subgroup S of relations is generated by

$$[[0 \hookrightarrow a]] = 1 \quad \forall a \in \mathcal{C}$$

$$[[f]] \cdot [[g]] = [[f \circ g]] \quad \text{if composable.}$$

Note $[a \to b] \in Mor\, Q\mathcal{C}$ corresponds to $0 \hookrightarrow a \to b \hookleftarrow 0$ which is a based loop.

If $[0 \hookrightarrow b' \hookrightarrow b] \in T \Rightarrow [[b' \hookrightarrow b]] = 1$ in $\pi_1(BQ\mathcal{C})$

since $[0 \hookrightarrow b' \hookrightarrow b] = [0 \hookrightarrow b'] \circ [b' \hookrightarrow b]$.

Therefore, $[[a \twoheadleftarrow b' \hookrightarrow b]] = [[a \twoheadleftarrow b']]$ in $\pi_1(BQ\mathcal{C})$.

Similarly $[0 \twoheadleftarrow a \twoheadleftarrow b] \in T \Rightarrow$ ~~$[[a \twoheadleftarrow b]]$~~

$\Rightarrow [[0 \twoheadleftarrow a]] \cdot [[a \twoheadleftarrow b]] = [[0 \twoheadleftarrow b]] \Rightarrow$

$[[a \twoheadleftarrow b]] = [[0 \twoheadleftarrow a]]^{-1} \cdot [[0 \twoheadleftarrow b]]$. in $\pi_1(BQ\mathcal{C})$.

$\Rightarrow \pi_1(BQ\mathcal{C})$ is generated by $[0 \twoheadleftarrow a]$ in $Q\mathcal{C}$.

let $[a \hookrightarrow b \twoheadrightarrow c] \in \mathcal{E}_{\mathcal{C}}$, we have in $Q\mathcal{C}$.

$[0 \hookrightarrow c] \circ [c \twoheadleftarrow b] = [0 \twoheadleftarrow a \hookrightarrow b]$, i.e.

$$\begin{tikzcd}
 & & a \arrow[dl, two heads] \arrow[dr, hook] & & \\
 & 0 \arrow[dl, equal] \arrow[d, hook] & & b \arrow[dll, two heads] \arrow[dr, equal] & \\
0 & c & & & b
\end{tikzcd}$$

Hence in $\pi_1(BQ\mathcal{C})$.

$$[[0 \twoheadleftarrow b]] = \cancel{[[0 \twoheadleftarrow c]]} \; [[0 \twoheadleftarrow c]] \cdot [[c \twoheadleftarrow b]]$$
$$= [[0 \twoheadleftarrow c]] \circ [[0 \twoheadleftarrow a \hookrightarrow b]]$$
$$= [[0 \twoheadleftarrow c]] \circ [[0 \twoheadleftarrow a]]$$

which is ~~the~~ an addition relation. Hence ~~this~~ every $[[f]] \circ [[g]] = [[f \circ g]]$ can be written in this form, and this is the only relation. We have.

$$K_0(\mathcal{C}) \simeq \pi_1(QB\mathcal{C}).$$

Higher K-theory

Define the K-space $K(\mathcal{C}) = \Omega BQ\mathcal{C}$.

Then $K_i(\mathcal{C}) = \pi_i(\Omega BQ\mathcal{C}) = \pi_{i+1}(BQ\mathcal{C})$ $i \geq 1$.

Note $F : \mathcal{C} \to \mathcal{D}$ exact functor, induces functors

$QF : Q\mathcal{C} \to Q\mathcal{D}$ $\quad BQF : BQ\mathcal{C} \to BQ\mathcal{D}$.

Hence homomorphic $BQF_i : K_i(\mathcal{C}) \to K_i(\mathcal{D})$.

We have natural functors

$$K: \underline{\text{Excat}} \longrightarrow \underline{\text{Spaces}}$$
$$\mathcal{C} \longmapsto K(\mathcal{C})$$

$$K_i . \underline{\text{Excat}} \longrightarrow \underline{\text{Ab}}$$
$$\mathcal{C} \longmapsto K_i(\mathcal{C}).$$

Ex A: unital associate ring, define $K_i(A) := K_i(\mathbb{P}(A))$ $i \geq 0$. This agrees with classical one.

THM $K_i(A) \cong \pi_i(K_0(A) \times BGL(A)^+)$ $\forall\ i \geq 0$.