

## Quillen's Higher algebraic K-theory

Ref: Quillen: Higher algebraic K-theory I.  
Homotopical algebra Sec 2.1-2.5.

Higher algebraic K-theory assigns any ring  $A$  a sequence of abelian groups  $K_i(A)$  in a functorial way. We define

$K_0(A) = \pi_0(K(A))$  for certain topological space  $K(A)$  constructed from  $A$ .

Step 1: Construct a small category  $\underline{K}(A)$

Step 2: Take the geometric realization  $B\underline{K}(A) = K(A)$ .

Recall  $K_2$ .

Steinberg group  $St_n(A)$  as a group generated by symbols  $x_{ij}(a)$  for  $1 \leq i \neq j \leq n$ ,  $a \in A$  modulo equivalence relations.

$$1) x_{ij}(a) \cdot x_{ij}(b) = x_{ij}(a+b)$$

$$2) [x_{ij}(a), x_{kl}(b)] = \begin{cases} 1 & j \neq k, i \neq l \\ x_{il}(ab) & j=k, i \neq l \\ x_{kj}(-ba) & j \neq k, i=l \end{cases}$$

Recall elementary matrices  $E_n = \langle e_{ij}(a) \rangle$  also satisfies above condition. Hence  $\exists$  natural map  $\varphi_n: St_n(A) \rightarrow GL_n(A)$ .

which factored as  $St_n(A) \rightarrow E_n(A) \hookrightarrow GL_n(A)$ .

Let  $St(A) = \text{colim } St_n(A)$ .

Recall  $K_1(A) = \text{coker}(\varphi)$  (since  $E(A) = [E(A), E(A)]$  by Whitehead.  $= [GL(A), GL(A)]$ )

Def:  $K_2 = \ker \varphi$

so  $\exists$  exact sequence of groups.

$$1 \rightarrow K_2(A) \rightarrow St(A) \xrightarrow{\varphi} GL(A) \rightarrow K_1(A) \rightarrow K_0(A) \rightarrow 1$$

THM (Steinberg)  $K_2$  is abelian. In fact  $K_2 = Z(St(A))$

Universal central extension.

Recall, a central extension of  $G$  by  $A$  is a s.e.s.

$$0 \rightarrow A \xrightarrow{i} \tilde{G} \xrightarrow{r} G \rightarrow 1$$

s.t.  $i(A) \subseteq Z(\tilde{G})$

Lemma:  $\exists$  a bijection  $\left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{central ext. of } G \text{ by } A \end{array} \right\} \longleftrightarrow H^2(G, A)$

Note: two central extensions are equivalent if  $\exists f: \tilde{G}_1 \rightarrow \tilde{G}_2$

s.t.

$$\begin{array}{ccccccc} 1 & \rightarrow & A & \rightarrow & \tilde{G}_1 & \rightarrow & G \rightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & B & & \tilde{G}_2 & & G \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & A & \rightarrow & \tilde{G}_2 & \rightarrow & G \rightarrow 1 \end{array}$$

$$1 \rightarrow A \xrightarrow{i_1} \tilde{G}_1 \xrightarrow{p_1} G \rightarrow 1$$

$$1 \rightarrow A \xrightarrow{i_2} \tilde{G}_2 \xrightarrow{p_2} G \rightarrow 1$$

commutes.

For a group  $G$ , define a category  $\underline{\text{Ext}}_c(G)$  with

$$\text{ob}(\underline{\text{Ext}}_c(G)) = \{ p: \tilde{G} \xrightarrow{p} G : \ker(p) \subseteq Z(\tilde{G}) \}$$

$$\text{Hom}_{\underline{\text{Ext}}_c(G)}(\tilde{G}_1, \tilde{G}_2) = \{ f: \tilde{G}_1 \rightarrow \tilde{G}_2 \mid p_2 \circ f = p_1 \}$$

Def: The universal central extension is ~~the~~ the initial object in  $\text{Ext}_c(G)$ .

THM: The universal central extension exists iff  $G$  is perfect.

THM (Recognition criterion)  $\tilde{G} \rightarrow G$  a central extension,

TFAE: 1)  $\tilde{G}$  is universal

2)  $\tilde{G}$  is perfect and every central extension of  $\tilde{G}$  splits

3)  $H_1(\tilde{G}, \mathbb{Z}) = H_2(\tilde{G}, \mathbb{Z}) = 0$ .

THM (Kervaire, Steinberg) The natural exact sequence  $1 \rightarrow K_2(A) \rightarrow St(A) \rightarrow E(A) \rightarrow 1$  is the universal central extension of  $E(A)$ .  $\therefore K_2(A) \cong H_2(E(A), \mathbb{Z})$

Def:  $K_3(A) = H_3(St(A), \mathbb{Z})$ .

Plus - construction.

Recall: A topological space  $X$  is acyclic if  $\tilde{H}_i(X, \mathbb{Z}) = 0$   
 $\forall i \in \mathbb{Z}$ .

If  $X$  is acyclic, we have: 1)  $X$  is connected

2)  $\pi_1(X)$  is perfect, i.e.  $\pi_1(X) = [\pi_1(X), \pi_1(X)]$

3)  $H_2(\pi_1(X), \mathbb{Z}) = 0$

Recall:  $Ff =$   
 $\{(e, r) \in (X, P)\}$   
 $f(e) = x(1),$   
 $\delta^{(1)} = x \}$ .

Def: Let  $X, Y$  be two based connected CW complex.

A map  $f: X \rightarrow Y$  is acyclic if its homotopy fiber  $Ff$  is acyclic.

Prop:  $X$  is ~~acyclic~~ acyclic  $\Leftrightarrow X \rightarrow *$  is acyclic.

Prop: Sp.  $f$  is acyclic, then 1)  $f_*: \pi_1(X) \rightarrow \pi_1(Y)$  is surjective

2)  $\text{Ker}(f_*)$  is a perfect normal subgroup of  $\pi_1(X)$ .

DEF Let  $X$  be a based CW-complex. Let  $N \trianglelefteq \pi_1(X)$  be a perfect normal subgroup of  $\pi_1(X)$ . A  $+$ -construction on  $X$  relative to  $N$  is an acyclic map  $f: X \rightarrow X_N^+$  s.t.  $\text{Ker}[f_*: \pi_1(X) \rightarrow \pi_1(X_N^+)] = N$ .

Quillen showed that  $+$ -construction exists and is universal among all maps  $g: X \rightarrow Y$  with  $g_*(N) = \{1\} \subset \pi_1(Y)$ .

$\Rightarrow X_N^+$  is unique (up to homotopy).

Every group  $G$  has a unique largest perfect group  $\Rightarrow$  denote this group  $P(G)$ , the perfect radical. Note,  $P(G) \subset G$  is normal.

Def: We call  $+$ -construction of on  $X$  w.r.t.  $P(\pi_1(X))$   
the plus construction:  $f: X \rightarrow X^+$ .

Higher  $K$ -groups.

Let  $A$  be a unital associative ring. Define  $K_i(A)$   
 $= \pi_i(BGL(A)^+)$ .

For  $i=1$ ,  $K_1(A) = \pi_1(BGL(A)^+) = \pi_1(BGL(A)) / P(\pi_1(BGL(A)))$   
(Whitehead lem)  $= GL(A) / [GL(A), GL(A)]$

Hence  $K_i: \underline{\text{Rings}} \rightarrow \underline{Ab}$

Def: The  $K$ -theory of  $A$  is the space.

$$K(A) = K_0 \times BGL(A)^+ = \coprod_{K_0} BGL(A)^+$$

Hence  $\pi_0(K(A)) = K_0(A)$

$$\pi_i(K(A)) = K_i(A) \quad i \geq 1$$

Prop. Quillen's def. of  $K_2(A)$  agrees with Milnor - Steinberg.

Pf: WTS  $\pi_2(BGL(A)^+) \cong H_2(E(A), \mathbb{Z})$ .

Lemma: Let  $N \trianglelefteq G$  perfect normal. Consider  $f: BG \rightarrow BG_N^+$ ,

let  $Ff$  be the homotopy fibre. Then

1)  $\pi_1(Ff)$  is isomorphic to the universal central extension  
of  $N$

$$2) \pi_2(BG_N^+) \cong H_2(N, \mathbb{Z})$$

Pf: We have homotopy fibration  $Ff \rightarrow BG \rightarrow BG_N^+$ .

Hence we have.

$$\begin{array}{ccccccccc} \dots & \pi_2(BG) & \rightarrow & \pi_2(BG_N^+) & \rightarrow & \pi_1(F) & \rightarrow & \pi_1(BG) & \rightarrow & \pi_1(BG_N^+) & \rightarrow & \pi_0(F) \\ & \parallel & & & & & & \parallel & & \parallel & & \parallel \\ & \{1\} & & & & & & G & & G/N & & \{1\}. \end{array}$$

Therefore, we get a s.e.s.

$$1 \rightarrow \pi_2(BG_N^+) \rightarrow \pi_1(F) \rightarrow N \rightarrow 1$$

Ret Whitehead  
Element of homotopy  
theory IV, 3.5

By a property of homotopy fibration,  $\pi_2(BG_N^+) \subset Z(\pi_1(F))$   
normal.

By previous observation,  $\pi_1(F)$  is a perfect subgroup.

$$\text{So } H_1(\pi_1(F), \mathbb{Z}) = 0 = H_2(\pi_1(F), \mathbb{Z}) = 0.$$

$\Rightarrow \pi_1(F) \cong$  universal central extension of  $N$ .

Thm

(Hopf). Consider a presentation.  $G \cong F/R$ ,  $\exists$  2 natural  
central extensions. 1.  $1 \rightarrow \frac{R}{[R, F]} \rightarrow \frac{F}{[F, F]} \rightarrow G \rightarrow 1$

$$2. \quad 1 \rightarrow \frac{R/[R, F]}{[R, F]} \rightarrow \frac{[F, F]}{[R, F]} \rightarrow [G, G] \rightarrow 1$$

$\exists$   $G$  perfect, the second one is the universal central extension  
and we have  $H_2(G, \mathbb{Z}) \cong \frac{R/[R, F]}{[R, F]}$ .

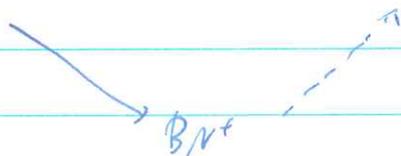
Hence, we have  $\pi_2(BG) \cong \pi_2(BG_N^+) \cong H_2(N, \mathbb{Z})$ .

Now let  $N = E(A) \subseteq G$   $U(A) = G$ , we get

$$K_2(A) = \pi_2(BGL(A)^+) = H_2(E(A), \mathbb{Z}). \quad \square$$

Lemma. Let  $N \trianglelefteq G$  be a perfect normal subgroup,  
 $f: BG \rightarrow BG_N^+$  the corresponding  $+$  construction,  
Then we have the factorization.

$$BN \longrightarrow BG \longrightarrow BG_n^+$$

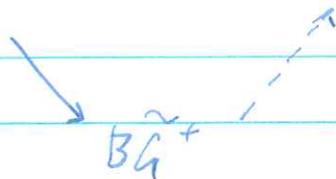


$\therefore BN^+$  is ~~homotopy~~ homotopy equiv. to the universal covering of  $BG_n^+$ . Therefore,  $\pi_i(BN^+) \cong \pi_i(BG_n^+)$   $i \geq 2$

Cor  $\forall$  ring  $A$ ,  $K_i(A) \cong \pi_i(BE(A)^+)$   $\forall i \geq 2$

Let  $1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1$  be the universal central extension of  $G$ . So  $G, \tilde{G}$  are perfect groups. Consider

$$BA \longrightarrow B\tilde{G} \longrightarrow BG \longrightarrow BG^+$$



We have a sequence  $BA \rightarrow B\tilde{G}^+ \rightarrow BG^+$

This is in fact a homotopy fibration sequence.

$$\pi_i(B\tilde{G}^+) = 0 \quad i = 0, 1, 2,$$

$$\pi_i(B\tilde{G}^+) = \pi_i(BG^+) \quad i \geq 3.$$

Cor.  $\forall i \geq 3$ ,  $\exists$  isomorphism  $K_i(A) \cong \pi_i(BS_+(A)^+)$

$$\text{Hence } K_3(A) = H_3(S_+(A), \mathbb{Z})$$

## Q - construction

### 1. Exact categories.

Def: An exact category is an additive category  $\mathcal{C}$  equipped w/ a class  $\mathcal{E}$  of diagram of the form  $(i, p) := [C' \xrightarrow{i} C \xrightarrow{p} C'']$ . We call  $i$  admissible monic, call  $p$  admissible epic.  $\mathcal{E}$  satisfies

E1) closed under isomorphisms.

E2) the set of admissible monics is closed under composition and pushouts along arbitrary morphisms. i.e.  $i: C \rightarrow C'$  admissible monic,  $f: C \rightarrow D$  arbitrary, then

$$\begin{array}{ccc} C & \xrightarrow{i} & C' \\ f \downarrow & & \downarrow \\ D & \xrightarrow{\tilde{i}} & D \times C' \end{array} \Rightarrow \tilde{i} \text{ also admissible monic.}$$

E3) the set of admissible epics is closed under compositions and pull backs. i.e.

$$\begin{array}{ccc} D \times C' & \xrightarrow{\tilde{p}} & D \\ \downarrow & & \downarrow f \\ C & \xrightarrow{p} & C' \end{array} \Rightarrow \tilde{p} \text{ admissible epic.}$$

E4) diagram of the form  $[C \hookrightarrow C \oplus D \rightarrow D] \in \mathcal{E}$

E5) if  $(i, p) \in \mathcal{E}$ ,  $i = \text{Ker}(p)$   $p = \text{Coker}(i)$

Def: A functor  $F: (\mathcal{C}, \mathcal{E}_C) \rightarrow (\mathcal{D}, \mathcal{E}_D)$  between



two exact categories are exact if  $F(\mathcal{C}, \mathcal{C}) = \mathcal{E}_0$ .

## 2. $K$ -group of an exact category.

Defn  $K_0(\mathcal{C}, \mathcal{E}) := \mathbb{Z}[\text{Iso}(\mathcal{C})] / ([C] = [C'] + [C''] \text{ for } [C'] \rightarrow C \rightarrow [C''] \text{ in } \mathcal{E})$

Ex. Let  $A$  be an abelian categories, then  $A$  is exact with  $\mathcal{E}$  being all short exact sequences.  $\Rightarrow$  canonical exact structure on  $A$ .

Note, there is another exact structure on  $A$ . Consider  $\mathcal{E} = \{ \text{all split exact sequences, i.e. } 0 \rightarrow C \rightarrow C \oplus D \rightarrow D \rightarrow 0 \}$ .  
In general,  $K_0(A, \mathcal{E}) \neq K_0(A, \mathcal{E}^{\text{split}})$ .

Ex. Let  $A$  be a unital ring. Consider  $\mathcal{C} = \text{HP}(A)$ , the category of f.g. proj. (right)  $A$ -modules. Take  $\mathcal{E}_{\text{HP}(A)}$  be all s.e.s.  $\mathcal{C}$  is not abelian, but exact. Then  $K_0(A) = K_0(\text{HP}(A), \mathcal{E}_{\text{HP}(A)})$ .

Ex.  $X$ : quasi-projective scheme, let  $\mathcal{C} = \text{Vect}(X)$  category of vector bundles on  $X$ .  $K^0(X) = K_0(\text{Vect}(X))$

Note  $K^0(X)$  is the Grothendich ring.

$K_0(X)$  is the Grothendich group defined by using coherent sheaves. In nice cases,  $K_0(X) \cong K^0(X)$ .

symmetric monoidal.

Ex. Let  $\mathcal{C}$  be a skeletally small  $\wedge$  category. We can define the relative groupoid  $\text{Iso}(\mathcal{C})$  associated to  $\mathcal{C}$ .

Let  $\mathcal{Z}_0(\mathcal{C}) = \pi_0(\underline{\text{Iso}}(\mathcal{C}))$ , this is an abelian monoid. Define a group  $K_0^{\oplus}(\mathcal{C})$  be the group completion of the monoid  $S = \underline{\text{Iso}}(\mathcal{C})$ . i.e.  $K_0^{\oplus}(\mathcal{C}) = S^{-1}S$ .

Let  $\mathcal{C}$  be an additive category, then it is naturally symmetric monoidal, with monoidal structure given by direct sum  $\oplus: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ . Consider K-group  $K_0^{\oplus}(\mathcal{C})$ . If  $\mathcal{C}$  is also exact, we have  $K_0(\mathcal{C}, \mathcal{E})$ . In general they are not isomorphic. In fact,  $K_0(\mathcal{C}, \mathcal{E})$  is naturally a quotient of  $K_0^{\oplus}(\mathcal{C})$ .

Lem. Let  $\mathcal{B}$  be an ~~exact~~ exact subcategory of  $\mathcal{C}$ , s.t.  $\mathcal{B}$  is closed under extension and cofinal. Then  $K_0(\mathcal{B})$  is naturally a subgroup of  $K_0(\mathcal{C})$ .

Cor  $K_0(\mathcal{C}) / K_0(\mathcal{B}) \cong K_0^{\oplus}(\mathcal{C}) / K_0^{\oplus}(\mathcal{B})$

### 3. Q-construction.

Let  $(\mathcal{C}, \mathcal{E})$  be skeletally small. Define a category  $\mathcal{Q}\mathcal{C}$  with:

1)  $\text{Ob}(\mathcal{Q}\mathcal{C}) = \text{Ob}(\mathcal{C})$ .  
 2)  $\text{Hom}_{\mathcal{Q}\mathcal{C}}(C, C') = \left\{ \begin{array}{c} \downarrow p \\ C \end{array} \begin{array}{c} \nearrow d \\ \downarrow i \\ C' \end{array} \right\} / \sim$  called 'roof'  
 where  $p$ : admissible epic  
 $i$ : admissible mono

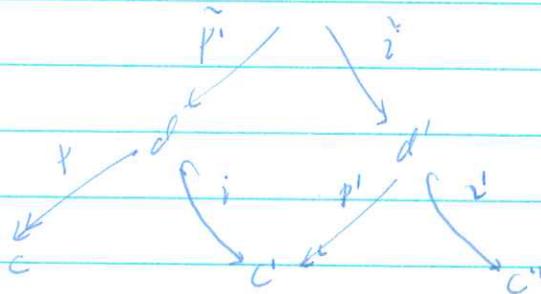
where  $\sim$  is: two roofs  $\begin{array}{c} C \xleftarrow{p} D \xrightarrow{i} C' \\ C \xleftarrow{p'} D' \xrightarrow{i'} C' \end{array}$

are equivalent if

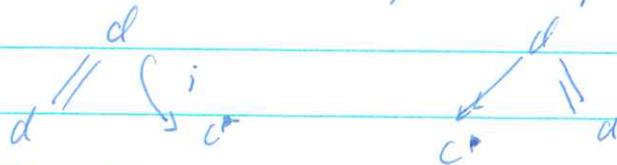
$\exists$  an isomorphism  $\varphi: d \xrightarrow{\sim} d'$  s.t.

$$\begin{array}{ccccc} c & \xleftarrow{p} & d & \xrightarrow{i} & c' \\ \parallel & & \cong \varphi & & \parallel \\ c & \xleftarrow{p'} & d' & \xrightarrow{i'} & c \end{array}$$

The composition of two roofs  $[c \xleftarrow{p} d \xrightarrow{i} c']$  and  $[c' \xleftarrow{p'} d' \xrightarrow{i'} c]$  is the equivalence class of  $d \times_{c'} d'$  where  $d \times_{c'} d'$  is pull-back of  $d'$  along  $i$ .



Rmk. There two distinguished type of roof.



which are exactly admissible monic/epic in  $\mathcal{C}$ .

We denote them by  $[d \xrightarrow{i} c]$  and  $[c \xleftarrow{p} d]$

Rmk.  $[c \xleftarrow{p} d \xrightarrow{i} c'] = [c \xleftarrow{p'} d'] \circ [d \xrightarrow{i} c']$

Def: Let  $c \in \text{Ob}(\mathcal{C})$ . A ~~subset~~ subobject of  $c$  is an equivalence class of monics  $d \hookrightarrow c$  s.t. two monics are equivalent if they factored through each other.  
 † subobject is admissible if it can be represented by admissible monic

Every morphism  $[c \leftarrow d \rightarrow c']$  defines a unique admissible monic  $d \hookrightarrow c'$ . Fix representatives for each subobject. Then a morphism  $c \rightarrow c'$  in  $\mathcal{Q}\mathcal{C}$  is given by an admissible subobject  $d \hookrightarrow c'$  in  $\mathcal{C}$  and an admissible epic  $d \twoheadrightarrow c$ . Hence,  $\text{Hom}_{\mathcal{Q}\mathcal{C}}(c, c') = \{ \text{admissible subobj. of } c' \text{ in } \mathcal{C} \}$ .

Also isomorphisms in  $\mathcal{Q}\mathcal{C}$  corresponds to iso morphisms in  $\mathcal{C}$ .

The  $K_0$  group via  $\mathcal{Q}$ -construction.

Let  $\mathcal{C}$  be skeletally small exact

Note 1)  $\mathcal{Q}\mathcal{C}$  is then skeletally small, so  $B\mathcal{Q}\mathcal{C}$  is well defined.

2)  $B\mathcal{Q}\mathcal{C}$  has an canonical base point given by  $0 \in \text{Ob}(\mathcal{C})$ .

3)  $B\mathcal{Q}\mathcal{C}$  is connected

THM  $K_0(\mathcal{C}) \cong \pi_1(B\mathcal{Q}\mathcal{C})$ , where  $[c] \in K_0(\mathcal{C})$  corresponds to  $[0 \hookrightarrow c \twoheadrightarrow 0] = [0 \hookrightarrow c] \circ [c \twoheadrightarrow 0]$ .

the class of

Pf: Let  $T$  be  $\times$  all morphisms  $[0 \hookrightarrow a] \in \text{Mor}(\mathcal{Q}\mathcal{C}) \Rightarrow T$  is a maximal tree. Then we have an presentation of  $\pi_1(B\mathcal{Q}\mathcal{C})$  (since  $\mathcal{Q}\mathcal{C}$  is small and connected):

$$\pi_1(B\mathcal{Q}\mathcal{C}) = \langle [ [\text{Mor } \mathcal{Q}\mathcal{C}] ] \rangle / S$$

where the normal subgroup  $S$  of relations is generated by.

$$[[0 \hookrightarrow a]] = 1 \quad \forall a \in \mathcal{C}$$

$$[[f]] \cdot [[g]] = [[f \circ g]] \quad \text{if composable.}$$

Note  $[a \rightarrow b] \in \text{Mor } \mathcal{Q}\mathcal{C}$  corresponds to  $0 \hookrightarrow a \rightarrow b \hookrightarrow 0$  which is a based loop.

If  $[0 \hookrightarrow b' \hookrightarrow b] \in T \Rightarrow [[b' \hookrightarrow b]] = 1$  in  $\pi_1(BQE)$

since  $[0 \hookrightarrow b' \hookrightarrow b] = [0 \hookrightarrow b'] \circ [b' \hookrightarrow b]$ .

Therefore,  $[[a \leftarrow b' \hookrightarrow b]] = [[a \leftarrow b']]$  in  $\pi_1(BQE)$ .

Similarly  $[0 \leftarrow a \leftarrow b] \in T \Rightarrow ~~[[a \leftarrow b]]~~$

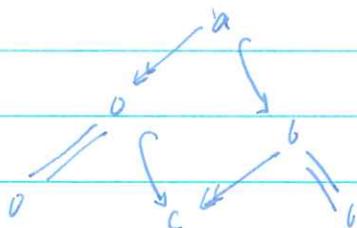
$\Rightarrow [[0 \leftarrow a]] \cdot [[a \leftarrow b]] = [[0 \leftarrow b]] \Rightarrow$

$[[a \leftarrow b]] = [[0 \leftarrow a]]^{-1} \cdot [[0 \leftarrow b]]$  in  $\pi_1(BQE)$ .

$\Rightarrow \pi_1(BQE)$  is generated by  $[0 \leftarrow a]$  in  $QE$ .

Let  $[a \hookrightarrow b \rightarrow c] \in QE$ , we have in  $QE$ .

$[0 \hookrightarrow c] \circ [c \leftarrow b] = [0 \leftarrow a \hookrightarrow b]$ , i.e.



Hence in  $\pi_1(BQE)$

$$[[0 \leftarrow b]] = ~~[[0 \leftarrow c]]~~ [[0 \leftarrow c]] \cdot [[c \leftarrow b]]$$

$$= [[0 \leftarrow c]] \circ [[0 \leftarrow a \hookrightarrow b]]$$

$$= [[0 \leftarrow c]] \circ [[0 \leftarrow a]]$$

which is ~~the~~ an additive relation. Hence ~~that~~ every  $[[f]] \circ [[g]]$

$= [[f \circ g]]$  can be written in this form. We have.

$K_0(\mathcal{C}) \cong \pi_1(QBE)$  and this is the only relations

## Higher K-theory

Define the K-span  $K(\mathcal{C}) = \Omega BQE$ .

Then  $K_i(\mathcal{C}) = \pi_i(\Omega BQE) = \pi_{i+1}(BQE)$   $i \geq 1$ .

Note  $F: \mathcal{C} \rightarrow \mathcal{D}$  exact functors, induces functors

$$KF: \mathcal{C} \rightarrow \mathcal{D} \quad BKF: BQE \rightarrow BQD$$

Hence homomorphic BQF's:  $K_i(\mathcal{L}) \rightarrow K_i(\mathcal{D})$ .

We have natural functors

$$K: \underline{\text{Excat}} \longrightarrow \underline{\text{Spaces}}$$
$$\mathcal{L} \mapsto K(\mathcal{L})$$

$$K_i: \underline{\text{Excat}} \longrightarrow \underline{\text{Ab}}$$
$$\mathcal{L} \mapsto K_i(\mathcal{L}).$$

Ex  $A$ : unital associative ring, define  $K_0(A) := K_0(\mathbb{P}(A))$   
 $i \geq 0$ . This agrees with classical one.

Thm  $K_i(A) \cong \pi_i(K_0(A) \times \text{BGL}(A^+)) \quad \forall i \geq 0.$